HISTORICAL EXERCISE SET.

1. As indicated earlier, Egyptian geometers used the formula A = (1/4)(a + c)(b + d) to calculate the area of any quadrilateral whose successive sides have lengths a, b, c, and d.

(a) Does this formula work for squares? For rectangles that are not squares?

This formula can be rewritten $\frac{a + c}{2} \frac{b+d}{2}$. Since in both rectangles (AAHH) and squares opposite sides have the same length we know that $\frac{a + c}{2} \frac{b+d}{2} = \frac{2a}{2}\frac{2b}{2} = ab$ where a and b are lengths of perpendicular sides. This is the modern formula for calculating the area of squares and rectangles.

(b) If you choose speciﬁc lengths for the sides of an isosceles trapezoid, how does the result compare to the actual area? Repeat for two other isosceles trapezoids. Do the same for three speciﬁc parallelograms.

An isosceles trapezoid has two same-length legs while the other two sides are opposite and parallel. Let us call the legs a and c while the other sides are b and d. So the formula the Egyptians used was $\frac{a + c}{2} \frac{b+d}{2} = \frac{2a}{2}\frac{b+d}{2} = a\frac{b+d}{2}$.

Isosceles 1:

Let a = 2, b = 3, c = 1. According to their formula, the area is 4 ($2\frac{1+3}{2}$). The actual area is calculated by dissecting the trapezoid into 3 separate sections, two right triangles with the legs of the trapezoid as their hypotenuses and a rectangle between them. First we know that the base and summit of the rectangle have length c, so there is length 2 left for legs of the right triangles. This gives these triangles legs of length 1 and $\sqrt{3}$ as determined by the Pythagorean theorem. This gives the rectangle height $sqrt{3}$ and therefore area $1(\sqrt{3})$ and each triangle will have area $\frac{\sqrt{3}}{2}$ Therefore the total area of the trapezoid is $2\sqrt{3}$ which is irrational and not equal to 4.

Isosceles 2: Let a=5, b=7, c=1. According to their formula the area is 20. Once again breaking the trapezoid into 2 right triangles and a rectangle we get a rectangle height (and triangle leg length) of 4. This gives the rectangle area 4 and the right triangle areas will add to 12 (each having area $6 = 3\frac{4}{2}$). This gives the total isosceles area to be 18.

Isosceles 3: Let a=5, b=9, c=3. According to their formula the area is 30. Once again breaking the trapezoid into 2 right triangles and a rectangle we get a rectangle height (and triangle leg length) of 4. This gives the rectangle area 12 and the right triangle areas will add to 12 (each having area $6 = 3\frac{4}{2}$). This gives the total isosceles area to be 24.

Parallelogram 1:

Let a=1, b=$\sqrt{2}$ and let one pair of opposite angles be 45 degree angles. Note that since opposite sides are of the same length, the Egyptian formula reduces to $ab$. Additionally, this cuts precisely into two right triangles each with hypotenuse $\sqrt{2}$ and legs length 1. This means that this parallelogram has area 1 (as each triangle has area .5), while the Egyptian formula would assign it area $\sqrt{2}$.

Parallelogram 2:

Let a=2, b=$\sqrt{2}$ and let one pair of opposite angles be 45 degree angles Additionally, this cuts precisely into a rectangle with length 1 and two right triangles each with hypotenuse $\sqrt{2}$ and legs length 1. The triangles again have area $.5$ each and the height of the rectangle, which is also a leg of the triangles is 1. Therefore the rectangle (square) has area 1. This means that this parallelogram has area 2 (as each triangle has area .5 and the square has area 1), while the Egyptian formula would assign it area $2\sqrt{2}$.

Parallelogram 3:

Let a=4, b=$5$ and let one pair of opposite angles be 45 degree angles Additionally, this cuts precisely into a rectangle with length 1 and two right triangles each with hypotenuse $5$ and each with a leg length 3. The triangles again have area $6$ (pythagorean triple triangle sides 3,4, and 5 thus area 6) each and the height of the rectangle, which is also a leg of the triangles is 4. Therefore the rectangle has area 4. This means that this parallelogram has area 16 (as each triangle has area 6 and the rectangle has area 4), while the Egyptian formula would assign it area $20$.

(c) Generalize your results for part (b).

The Egyptian formula for the area of a isosceles trapezoid gives an area that is too large unless the isosceles trapezoid is a rectangle. A formula for the area of the trapezoid would be the average of the opposite sides of different length (base and summit as opposed to the legs) times the height of the rectangle. The Egyptian formula can be written as the product of the averages of opposite sides, which shares a part with the other formula I have provided. Since the legs have the same length, the part of the product that they produce is an overestimate of the height except in the case where the trapezoid is a rectangle. Therefore $A = h\frac{b + d}{2} \leq a\frac{b+d}{2}$ since $h \leq a$.

The Egyptian formula for the area of a parallelogram would only work if it is indeed a rectangle. As you change the angles on the parallelogram, the area decreases but the Egyptian formula does not show this. This can be demonstrated by reducing a pair of opposite angles from 90 to 0. As this happens, adjacent sides come closer to becoming a straight line (when the angle between them reaches 0). When this happens, the area of the parallelogram reaches 0. Therefore the area of the parallelogram according to the Egyptians is always greater than or equal to our modern calculations.

2. If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, Babylonian geometers approximated the length of the hypotenuse by the formula

c = b + (a^ 2/2b).

(a) How does this approximation compare to the actual result when a = 3 and b = 4? When a = 5 and b = 12? When a = 12 and b = 5?

Case 1: a = 3, b = 4. When these are the legs this forms a Pythagorean triple with c= 5. Using the Babylonian formula, we get $c= 4 + (9/8) = 5.125$. The actual hypotenuse is 5 showing that this is an overestimation.

Case 2: a = 5 and b = 12 . These legs form a Pythagorean triple with c = 13. Using the Babylonian formula yields $c = 12 + (25/24) = 13 + (1/24)$

Case 3: a = 12 and b = 5. These legs again form the Pythagorean triple with c = 13. Using the

(b) Give an algebraic argument demonstrating that this formula results in an approximation that is too large.

3. The following was translated from a Babylonian tablet created about 2600 B.C. Explain what it means.

60 is the circumference, 2 is the perpendicular, ﬁnd the chord. Double 2 and get 4, do you see?

Take 4 from 20 and get 16. Square 20, and you get 400. Square 16, and you get 256. Take 256

from 400 and you get 144. Find the square root of 144. 12, the square root, is the chord. This

is the procedure.

4. The Moscow Papyrus (c. 1850 B.C.) contains the following problem:

If you are told: A truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the

top. You are to square this 4, result 16. You are to double 4, result 8. You are to square 2,

result 4. You are to add the 16, the 8, and the 4, result 28. You are to take one third of 6,

result 2. You are to take 28 twice, result 56. See, it is 56. You will ﬁnd it right.

Show that this is a special case of the general formula, V = (1/3)h(a^2 + ab + b

2

), for the volume of the

frustum of a pyramid whose bases are squares, whose sides are a and b, respectively, and whose height is

h. Also, use calculus to derive the formula mentioned above.

5. An Egyptian document, the Rhind Papyrus (c. 1650 B.C.), states that the area of a circle can be

determined by ﬁnding the area of a square whose side is 8/9 of the diameter of the circle. Is this correct?

What value of π is implied by this technique?

6. It is said that Thales indirectly measured the distance from a point on shore to a ship at sea using the

equivalent of angle-side-angle (ASA) triangle congruence theorem. Make a diagram that could be used to

accomplish this feat.

7. Eratosthenes (c. 275 B.C.), a scholar and librarian at the University at Alexandria, is credited with

calculating the circumference of the earth using the following method: Eratosthenes observed that on the

summer solstice the sun was directly overhead at noon in Syene (the present site of Aswan), while at the

same time in Alexandria, which was due north, the rays of the sun were inclined 7

◦

12’, thus indicating

that Alexandria was 7

◦

12’ north of Syene along the earth’s surface. Using the known distance between the

two cities of 5000 stades (approximately 530 miles), he was able to approximate the circumference of the

earth. Make a diagram that depicts this method and calculate the circumference in stades and in miles.

How does this result compare to present-day estimates?